

# All static spherically symmetric perfect fluid solutions of Einstein's Equations

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An algorithm based on the choice of a single monotone function (subject to boundary conditions) is presented which generates all regular static spherically symmetric perfect fluid solutions of Einstein's equations. For physically relevant solutions the generating functions must be restricted by non-trivial integral-differential inequalities. Nonetheless, the algorithm is demonstrated here by the construction of an infinite number of previously unknown physically interesting exact solutions.

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Exact solutions of Einstein's field equations provide a route to the physical understanding (and discovery) of relativistic phenomena, a convenient basis from which perturbation methods can proceed and a check on numerical approximations. Here we look at static spherically symmetric perfect fluid solutions. Unfortunately, even for this simple type, very few solutions are in fact known, and of these few pass even elementary tests of physical relevance [1]. In this paper, an algorithm based on the choice of a single monotone function (subject to boundary conditions) is presented which generates all regular static spherically symmetric perfect fluid solutions of Einstein's equations. We are interested only in physically relevant solutions here and so the algorithm must be supplemented by physical considerations [2]. These additional conditions limit the generating functions allowed by way of non-trivial integral-differential inequalities. The details of how to choose physically relevant generating functions (beyond trial and error) are, at present, not known. Nonetheless, the robustness of the algorithm is demonstrated here by the construction of an infinite number of previously unknown physically interesting exact solutions.

To set the notation, consider a spherically symmetric spacetime  $\mathcal{M}$  [3]

$$ds_{\mathcal{M}}^2 = ds_{\Sigma}^2 + R^2 d\Omega^2 \quad (1)$$

where  $d\Omega^2$  is the metric of a unit sphere ( $d\theta^2 + \sin^2(\theta)d\phi^2$ ) and  $R = R(x^1, x^2)$  where the coordinates on the Lorentzian two space  $\Sigma$  are labelled as  $x^1$  and  $x^2$ . Consider a flow (a congruence of unit timelike vectors  $u^\alpha$ ) tangent to an open region of  $\Sigma$  and write  $n^\alpha$  as the normal to  $u^\alpha$  in the tangent space of  $\Sigma$ . Both  $u^\alpha$  and  $n^\alpha$  are uniquely determined. We suppose that (1) is generated by a fluid subject to the condition  $G_{\alpha}^{\beta} u^{\alpha} n_{\beta} = 0$  where  $G_{\alpha}^{\beta}$  is the Einstein tensor (see [4]). Let  $G \equiv G_{\alpha}^{\alpha}$ ,  $G1 \equiv G_{\alpha}^{\beta} u^{\alpha} u_{\beta}$  and  $G2 \equiv G_{\alpha}^{\beta} n^{\alpha} n_{\beta}$ . In the static case it follows that the flow is shear free and that

$$G + G1 = 3G2 \quad (2)$$

is a necessary and sufficient condition for (1) to represent a perfect fluid [5].

First consider  $\mathcal{M}$  in "curvature" coordinates,

$$ds_{\mathcal{M}}^2 = \frac{dr^2}{1 - \frac{2m(r)}{r}} + r^2 d\Omega^2 - e^{2\Phi(r)} dt^2. \quad (3)$$

Writing out (2) [6] we obtain an expression involving  $\Phi(r)$  and  $m(r)$  with derivatives to order two in  $\Phi(r)$  and to order one in  $m(r)$ . Viewing (2) as a differential equation in  $\Phi(r)$ , given  $m(r)$ , we obtain a Riccati equation in the first derivative of  $\Phi(r)$ . However, viewing (2) as a differential equation in  $m(r)$ , given  $\Phi(r)$ , we obtain a linear equation of first order [7]. As a consequence, we have the following algorithm for constructing all possible spherically symmetric perfect fluid solutions of Einstein's equations:

Given  $\Phi(r)$  (sufficiently smooth and subject to boundary conditions explained below)

$$m(r) = \frac{\int b(r) e^{\int a(r) dr} dr + \mathcal{C}}{e^{\int a(r) dr}} \quad (4)$$

where

$$a(r) \equiv \frac{2r^2(\Phi''(r) + \Phi'(r)^2) - 3r\Phi'(r) - 3}{r(r\Phi'(r) + 1)} \quad (5)$$

and

$$b(r) \equiv \frac{r(r(\Phi''(r) + \Phi'(r)^2) - \Phi'(r))}{r\Phi'(r) + 1} \quad (6)$$

with  $' \equiv \frac{d}{dr}$  and  $\mathcal{C}$  a constant. The generating function associated with any known solution is of course immediately obvious following the algorithm.

Interior boundary conditions on  $\Phi(r)$  are set by the requirement that all invariants polynomial in the Riemann tensor are finite at the origin. In this case there are but three independent invariants [8] and these are expressed here in terms of the physical variables; the energy density

$$\rho = \frac{G1}{8\pi} = \frac{m'(r)}{4\pi r^2} \geq 0 \quad (7)$$

and the isotropic pressure

$$p = \frac{G2}{8\pi} = \frac{r\Phi'(r)(r - 2m(r)) - m(r)}{4\pi r^3} \geq 0. \quad (8)$$

Note that the inequalities in (7) and (8) are to be viewed as imposed restrictions on  $\Phi(r)$ . At the centre of symmetry ( $r = 0$ ) the regularity of the Ricci invariants requires that  $\rho(0)$  and  $p(0)$  be finite. The regularity of the Weyl invariant requires that  $m(r)$  is  $C^3$  at  $r = 0$  with  $m(0) = m(0)' = m(0)'' = 0$  and  $m(0)''' = 8\pi\rho(0)$  [9]. In summary, for a static spherically symmetric perfect fluid, finite  $\rho(0)$  and  $p(0)$  guarantees the regularity of all Riemann invariants at the centre of symmetry.  $\Phi(0)$  is a finite constant (set by the scale of  $t$ ) and it follows from (8) that  $\Phi'(0) = 0$  and  $\Phi''(0) = \frac{4\pi}{3}(3p(0) + \rho(0)) > 0$ . Since  $\rho \geq 0$  and continuous and since  $p(0) > 0$  and finite it follows from (2) that  $r > 2m(r)$  [10]. With  $r > 2m(r)$  for  $r > 0$  it also follows from (8) for  $p(r) > 0$  that  $\Phi'(r) \neq 0$  for  $r > 0$ . As a result, the source function  $\Phi(r)$  must be a monotone increasing function with a regular minimum at  $r = 0$ . Exterior boundary conditions on  $\Phi(r)$  exist only for isolated spheres and these conditions are set by junction conditions [11]. The necessary and sufficient condition that  $\mathcal{M}$  have a regular boundary surface with a Schwarzschild vacuum exterior at  $r = R > 0$  is given by  $p(r = R) = 0$ . Setting  $m(r = R) \equiv M$  it follows that  $\Phi'(r = R) = \frac{M}{R(R-2M)}$ .

Each source function  $\Phi(r)$  which is a monotone increasing function with a regular minima at  $r = 0$  necessarily gives, via (4), a static spherically symmetric perfect fluid solution of Einstein's equations which is regular at  $r = 0$ . Exact solutions, in the present context, can be viewed as those for which (4) can be evaluated without recourse to numerical methods. The number of source functions  $\Phi(r)$  for which (4) can be evaluated exactly is infinite. It should be noted, however, that the generation of an exact solution does not necessarily mean that the equation  $p(r = R) = 0$  can be solved exactly. The algorithm presented here is now demonstrated by the construction of an infinite number of previously unknown but physically interesting exact solutions of Einstein's equations.

Let

$$\Phi(r) = \frac{1}{2} N \ln(1 + \frac{r^2}{\alpha}) \quad (9)$$

where  $N$  is an integer  $\geq 1$  and  $\alpha$  is a constant  $> 0$ . The function (9) is monotone increasing with a regular minimum at  $r = 0$ . With the source function (9), (4) can be evaluated exactly for any  $N$ . Whereas (9) generates a "class" of solutions, the metric (in particular  $m(r)$ ) looks quite distinct, and the physical properties are quite distinct, for each value of  $N$ . Previously, only for  $N = 1, \dots, 5$  were solutions known, having been arrived at by various methods, and one solution which is the first term in the Taylor expansion of (9) [12]. (These

solutions, with  $N = 1, \dots, 5$ , in fact constitute half of all the previously known physically interesting solutions (of this type) in curvature coordinates.) For  $N \geq 5$  the solutions are acceptable on physical grounds and even exhibit a monotonically decreasing subluminal adiabatic sound speed [13].

It is, perhaps, worth noting here that the foregoing discussion in curvature coordinates can be transformed directly into Bondi radiation coordinates [14].

Now consider "isotropic coordinates"

$$ds_{\mathcal{M}}^2 = e^{2B(r)}(dr^2 + r^2 d\Omega^2) - e^{2(\Psi(r)-B(r))} dt^2. \quad (10)$$

Unlike curvature coordinates, the isotropic form (10) does not offer an immediate invariant physical interpretation of the functions  $\Psi(r)$  or  $B(r)$  [15]. However, as we now show, the coordinates offer a simplified algorithm for constructing perfect fluid solutions. Writing out (2) we now obtain an expression involving  $\Psi(r)$  and  $B(r)$  with derivatives to order two in  $\Psi(r)$  and to order one in  $B(r)$ . Viewing (2) as a differential equation in  $\Psi(r)$ , given  $B(r)$ , we again obtain a Riccati equation in the first derivative of  $\Psi(r)$ . However, viewing (2) as a differential equation in  $B(r)$ , given  $\Psi(r)$ , we obtain an equation solvable simply by quadrature. As a consequence, we have the following simplified algorithm for constructing all possible spherically symmetric perfect fluid solutions of Einstein's equations in isotropic coordinates:

Given  $\Psi(r)$  (sufficiently smooth and subject to boundary conditions explained below)

$$B(r) = \Psi(r) + \int c(r) dr + \mathcal{C} \quad (11)$$

where

$$c(r) \equiv \frac{\epsilon}{\sqrt{2}} \sqrt{(\Psi'(r))^2 - \Psi''(r) + \Psi'(r)/r} \quad (12)$$

with  $\epsilon = \pm 1$ ,  $' \equiv \frac{d}{dr}$  and  $\mathcal{C}$  a constant. Recently, Rahman and Visser [16] have also presented an algorithm for constructing spherically symmetric perfect fluid solutions in isotropic coordinates. The source function  $\Psi(r)$  used here is related to the source function  $z(r)$  used by Rahman and Visser as follows:

$$\Psi(r) = 2 \int \frac{rz(r)}{1 - z(r)r^2} dr. \quad (13)$$

The two algorithms differ fundamentally in the sense that only one integration is used in the present procedure as opposed to two distinct integrations used in the Rahman-Visser procedure. The Rahman-Visser procedure was motivated by the requirement that metric be manifestly real ab initio. The reality of the integral (11) is discussed below.

Interior boundary conditions on  $\Psi(r)$  are set exactly as in the case of curvature coordinates. We now have the energy density and pressure in the form

$$\rho = \frac{G1}{8\pi} = \frac{-1}{8\pi e^{2B(r)}} (2B''(r) + \frac{4B'(r)}{r} + (B'(r))^2) \geq 0 \quad (14)$$

and

$$p = \frac{G2}{8\pi} = \frac{-1}{8\pi e^{2B(r)}} (-B'(r)\Psi'(r) + (B'(r))^2 - 2\frac{\Psi'(r)}{r}) \geq 0. \quad (15)$$

$\Psi(0)$  is a finite constant (set by the scale of  $t$ ) and it follows from (15) that  $\Psi'(0) = 0$  and from (11) that  $B'(0) = 0$ . With  $p(r) \geq 0$  it follows that the source function  $\Psi(r)$  must be a monotone increasing function with a regular minimum at  $r = 0$  and  $\Psi''(0) = 4\pi e^{2B(0)}p(0)$ . Exterior boundary conditions on  $\Psi(r)$  are set as in curvature coordinates. Regularity of  $\rho(0)$  requires  $B'(0) = 0$  and with  $\rho(r) \geq 0$  it follows that  $B(r)$  must be a monotone decreasing function with a regular maximum at  $r = 0$  and  $B''(0) = -4\pi e^{2B(0)}\rho(0)$ . The limits  $-\frac{2}{r} < B'(r) < 0$  guarantee the positivity of the effective gravitational mass. To examine the reality of the metric consider the function  $F(r) \equiv (\Psi'(r))^2 - \Psi''(r) + \Psi'(r)/r$ . Now  $F(0) = 0$ ,  $F'(0) = 0$  and  $F''(0) > 0$  so  $F(r)$  has a local minimum at  $r = 0$ . Now suppose that  $F(r) = 0$  for  $r > 0$ . Then condition (2) requires  $B' = \Psi'$  so we have already passed through a region with  $\rho < 0$  before the reality of the metric breaks down (in agreement with known theorems [10]).

In parallel to the algorithm in curvature coordinates, each source function  $\Psi(r)$  which is a smooth monotone increasing function with a regular minima at  $r = 0$  necessarily gives, via (11), a static spherically symmetric perfect fluid solution of Einstein's equations which is regular at  $r = 0$ . Exact solutions are again those for which (11) can be evaluated without recourse to numerical methods. Physical considerations must guide the choice of  $\Psi(r)$ . In isotropic coordinates, ratios of invariants and differential invariants can be obtained directly from the source function  $\Psi(r)$  via differentiation. You do not need  $B(r)$  and in particular you do not need to integrate. For example, the functions  $p(r)/\rho(r)$  and  $p'(r)/\rho'(r)$  follow directly without integration. Of course, neither  $p(r)$  nor  $\rho(r)$  follow without integration. In curvature coordinates you cannot get these ratios without integration, starting from the source function  $\Phi(r)$ .

To demonstrate the algorithm in isotropic coordinates

let

$$\Psi(r) = \alpha \ln \frac{f(r)}{g(r)} \quad (16)$$

where  $\alpha$  is a constant  $> 0$ . Of course it is not difficult to find functions  $f(r)$  and  $g(r)$  so that (16) is monotone increasing with a regular minimum at  $r = 0$ . Nor indeed is it difficult to find such functions for which  $B(r)$  can be evaluated exactly. For example, let  $g(r) = (\delta + \epsilon r^2)^\zeta$  and  $f(r) = \delta^\zeta + \gamma r^2$  with  $\delta, \epsilon, \gamma$  and  $\zeta$  constants such that  $\delta > 0$  and  $\delta^{1-\zeta}\gamma > \zeta\epsilon$ . This class of solutions includes a number of known solutions including the Schwarzschild interior solution and the Rahman-Visser general quadratic ansatz. Any solution in isotropic coordinates can be immediately recovered and generalized following the algorithm presented [17].

An algorithm based on the choice of a single monotone function (subject to boundary conditions) has been presented which generates all regular static spherically symmetric perfect fluid solutions of Einstein's equations. In all cases the choice of generating function must be guided by physical considerations. These additional conditions limit the generating functions allowed by way of non-trivial integral-differential inequalities. The details of how to choose physically relevant generating functions (beyond trial and error) are, at present, not known. Moreover, the resultant equation of state is a byproduct of the algorithm and can not be set a priori. Despite these reservations, the algorithm has been demonstrated by the construction of an infinite number of previously unknown physically interesting exact solutions [18]. It is a curious fact of history that over half a century ago Max Wyman [19] pointed out that the algorithm presented here was possible and yet, despite the voluminous literature on the subject [1], the algorithm appears not have been followed up.

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[1] See M. S. R. Delgaty and K. Lake, Computer Physics Communications **115**, 395 (1998) (gr-qc/9809013).

[2] The conditions used in [1] were: (i) isotropy of the pressure (otherwise any metric is a "solution"), (ii) regularity at the origin, (iii) positivity of the pressure and energy

density at the origin, (iv) vanishing of the pressure at a finite boundary, (v) monotone decrease of the energy density to the boundary and (vi) subluminal adiabatic sound speed. In addition to these, a monotone decrease in the subluminal adiabatic sound speed is desirable.

- [3] We use geometrical units throughout. The “curvature coordinates” used in (3) have the advantage that the metric functions have a clear invariant physical interpretation (but see also [14] below). The function  $m(r)$  is the effective gravitational mass. See W. C. Hernandez and C. W. Misner, *Astrophys. J.* **143**, 452 (1965), E. Poisson and W. Israel, *Phys. Rev D* **41**, 1796 (1990), T. Zannias, *Phys. Rev. D* **41**, 3252 (1990) and S. Hayward *Phys. Rev. D* **53**, 1938 (1996) (gr-qc/9408002). Whereas  $\Phi(r)$  is (in

the weak field limit) the “Newtonian” potential,  $re^{-\Phi(r)}$  is the effective potential for null geodesics (see, for example, M. Ishak, L. Chamandy, N. Neary and K. Lake, *Phys. Rev. D* **64** 024005 (2001) (gr-qc/0007073)).

- [4] gr-qc/0209063  
 [5] One can take the view that the Tolman-Oppenheimer-Volkoff equation is a consequence of the invariant statement (2).  
 [6] Explicitly, condition (2) in the static case in curvature coordinates reduces to the Walker pressure isotropy condition  $G_r^r = G_\theta^\theta$  (see A. G. Walker, *Quarterly Journal of Mathematics*, **6**, 81 (1935)) which is

$$\left(\frac{d^2}{dr^2}\Phi(r) + \left(\frac{d}{dr}\Phi(r)\right)^2\right)r^2(r - 2m(r)) - r\left(\frac{d}{dr}\Phi(r)\right)\left(\left(\frac{d}{dr}m(r)\right)r + r - 3m(r)\right) + 3m(r) - \left(\frac{d}{dr}m(r)\right)r = 0.$$

- [7] The problem has also been reduced to a linear equation of first order by A. S. Berger, R. Hojman and J. Santamarina, *J. M. P.* **28**, 2949 (1987). Recently G. Fodor (gr-qc/0011040) has reduced the problem to an algebraic one with integration required only for one metric function but not the physical variables  $\rho$  and  $p$ .  
 [8] D. Pollney, N. Pelavas, P. Musgrave and K. Lake, *Computer Physics Communications* **115**, 381 (1998) (gr-qc/9809012).  
 [9] It follows from (2) and (3) that the necessary and sufficient condition for conformal flatness for  $r > 0$  is given by  $m(r) = cr^3$ , which gives, uniquely, the Schwarzschild interior solution. See also H. A. Buchdahl, *A. J. P.* **39**, 158 (1971).  
 [10] See T. W. Baumgarte and A. D. Rendall, *Class. Quant. Grav.* **10**, 327 (1993) and also M. Mars, M. Mercè Martín-Prats and J. M. M. Senovilla, *Phys. Lett A* **218**, 147 (1996) (gr-qc/0202003).  
 [11] See, for example, P. Musgrave and K. Lake, *Class. Quantum Grav.* **13**, 1885 (1996) (gr-qc/9510052). At an interior boundary surface  $p$ , but not  $\rho$ , must be continuous. Discontinuities in  $\rho$  are associated with phase transitions which we do not considered here. For a discussion of interior phase transitions see, for example, L. Lindblom, *Phys. Rev. D* **58**, 024008 (1998).  
 [12] In terms of the classification given in [1] the solutions are **Tolman IV** for  $N = 1$ , **Heint IIa** for  $N = 3$ , **Durg IV** for  $N = 4$  and **Durg V=D-F** for  $N = 5$ . If  $\Phi(r)$  is taken to be the first term in the Taylor expansion of (9), the solution is known as **Kuch2 III**. The case  $N = 2$  gives  $m(r) = Cr^3/(3r^2 + \alpha)^{2/3}$  which is usually dismissed under the erroneous assumption  $C = 0$ .  
 [13] N. Neary, J. Lattimer and K. Lake (in preparation).  
 [14] These were first discussed (in the spherically symmetric case) by H. Bondi, *Proc. R. Soc. London A* **281**, 39 (1964) and are a generalization of the well known Eddington-Finkelstein coordinates for the Schwarzschild vacuum.

The algorithm presented is equally at home in curvature and radiation coordinates. Writing

$$dv = dt \pm \frac{e^{-\Phi(r)}}{\sqrt{1 - 2m(r)/r}} dr,$$

+ for advanced (ingoing)  $v$  and – for retarded (outgoing)  $v$  it follows that (3) takes the form

$$ds^2 = \pm 2 \frac{e^{\Phi(r)}}{\sqrt{1 - 2m(r)/r}} dv dr + r^2 d\Omega^2 - e^{2\Phi(r)} dv^2.$$

The form of condition (2) (given above in [6]) remains unchanged as do the functional forms and physical meanings of  $\Phi$ ,  $m$ ,  $\rho$  and  $p$ .

- [15] We proceed here in isotropic coordinates ab initio without coordinate transformations. Now  $re^{2B(r) - \Psi(r)}$  is the effective potential for null geodesics and the effective gravitational mass is given by  $m(r) = -(B'(r)(B'(r)r + 2)e^{B(r)}r^2)/2$  where  $' \equiv d/dr$ .  
 [16] S. Rahman and M. Visser, *Class. Quant. Grav.* **19**, 935 (2002) (gr-qc/0103065).  
 [17] In the terminology of [1], as regards physically interesting solutions, the **P-S2** solution follows from the stated form of  $\Psi(r)$ . Similarly, the choices  $g(r) = \cosh(\beta + \gamma r^2)$  and  $f(r) = \sinh(\beta + \gamma r^2)$  with  $\beta$  and  $\gamma$  positive constants immediately gives **Gold III**.  
 [18] It is also of interest to note that seven of the eleven previously known solutions of this type are special cases resulting from the two generating functions considered here.  
 [19] M. Wyman, *Phys. Rev* **75**, 1930 (1949).  
 [20] This is a package which runs within Maple. It is entirely distinct from packages distributed with Maple and must be obtained independently. The GRTensorII software and documentation is distributed freely on the World-Wide-Web from the address <http://grtensor.org>